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LUSTERNIK–SCHNIRELMANN CATEGORY OF 3-MANIFOLDS

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§1. INTRODUCTION

The Lusternik–Schnirelmann category $catX$ of a space X is the smallest number of sets, open and contractible in X, needed to cover X. We prove in this paper (Corollary 4.2) that, for a closed 3-manifold M^3, $catM^3$ depends only on $\pi_1 M^3$: it is 2 if $\pi_1 M^3$ is trivial, 3 if it is free nontrivial, and 4 if it is not free. Recall that a closed 3-manifold M^3 has a free fundamental group if and only if each prime summand of M^3 is a homotopy sphere or an S^2-bundle over S^1 [13; chapter 5]. Thus, question 12 in [26] has an affirmative answer.

The smallest number of open cells needed to cover M^3 is denoted by $C(M^3)$. This invariant has been calculated by Hempel and McMillan [14]. Endowing M^3 with a differentiable structure (and there is essentially only one), if f is a smooth function on M^3, the number of critical points of f is called $\mu_{M^3}(f)$. The minimal value of $\mu_{M^3}(f)$ for a fixed M^3, over all f, is denoted by $F(M^3)$. This invariant has been calculated by Takens [33]. From their results it follows that $C(M^3) = F(M^3)$. We will show that $catM^3 = C(M^3)$ if and only if M^3 contains no fake cells or $\pi_1 M^3$ is not free and so, modulo the Poincaré conjecture, the three invariants cat, C and F coincide on closed 3-manifolds.

A subset U of a space X is π_1-contractible if every loop in U is contractible in X. We denote by $cat_{\pi_1} X$ the smallest number of open π_1-contractible sets needed to cover X. This invariant was defined by Fox [7], who denoted it by $h_1 cat X$, and has been studied, for example, in [5] and [9]. For a closed 3-manifold M^3 we prove that $cat_{\pi_1} M^3$, again, depends only on $\pi_1 M^3$: it is 1 if $\pi_1 M^3$ is trivial, 2 if it is free nontrivial, and 4 if it is not free (Corollary 4.2).

To prove our results we use a theorem of Ganea–Eilenberg (see also Proposition 2.1) and the natural homomorphism $H_3(M^3; A) \to H_3(\pi_1 M^3; A)$. The homology of (possibly nonorientable) 3-manifold groups is calculated in §3.

With similar methods we prove that if M^n is properly covered by a homotopy n-sphere then $catM^n = cat_{\pi_1} M^n = n + 1$, thus giving a new proof of a theorem of Krasnoselski ([18]). This proof, as well as that in [23], is algebraic-topological as requested by James in [16]. We have learned that S. Husseini also has an unpublished proof of Krasnoselski's theorem.

In §2 we give an H_1 version and a π_1 version of a theorem of Eilenberg and Ganea ([5, Prop. 3], [9, Prop. 1.18]). Our proof of the π_1 version seems to us more direct than Ganea's proof. Also we omit the condition of semilocal 1-connectedness. The H_1 version will be used in a subsequent paper. We thank Mónica Clapp for pointing out the existence of [9].

In §3 we study the homomorphism $H_3(M^3; A) \to H_3(\pi_1 M^3; A)$. If M^3 is irreducible with infinite fundamental group, the classifying space of $\pi_1 M^3$ is obtained by attaching to M^3 copies of $\mathbf{P}^\infty$ along a maximal family of two sided $\mathbf{P}^2$'s no two of which cobound a homotopy $\mathbf{P}^2 \times I$. This allows one to calculate the homology of $\pi_1 M^3$ and the image of $H_3(M^3; A) \to H_3(\pi_1 M^3; A)$.

In §4 we prove the main theorem (Theorem 4.1), and calculate $catM^3$ and $cat_{\pi_1} M^3$ for any closed 3-manifold M^3 (Corollary 4.2). We also relate $cat(M)$ to two other invariants $C(M)$ and $N_0(M)$, the smallest number of balls and the smallest numbers of charts needed to cover M. It is also pointed out that if M^3 is obtained by non trivial surgery on a non trivial knot k, then $C(M^3) = 4$ and, if $\pi_1 M^3 \neq 1$, $catM^3 = 4$. We prove that, if M^3 is a closed 3-manifold, $cat(M^3\text{-}point) = cat(M^3) - 1$; this answers affirmatively, for 3-manifolds, question 10 in [26] (see also Conjecture 7.2 in [27]).

In §5 we apply our methods to prove that if M^n is properly covered by S^n then $catM^n = cat_{\pi_1} M^n = n + 1$.

§2. PRELIMINARIES

The *Lusternik–Schnirelmann category* $catX$ of a topological space is the least integer n such that X can be covered by n open sets $U_1, \ldots, U_n$ such that each U_i is contractible in X. If no such integer exists then $catX = \infty$.

More generally, let F: $Top \to \mathscr{C}$ be a functor on the category of topological spaces. A subset U of a space X is F-contractible (in X) if $F(i)$ is a constant morphism ([15, Definition 8.2]) where $i: U \to X$ is the inclusion. We write $cat_F X = n$ if n is the least integer such that X can be covered by n open subsets $U_1, \ldots, U_n$ such that each U_i is F-contractible. Again, if no such integer exists then $cat_F X = \infty$.

If F: $Top \to hTop$ is the natural functor to the homotopy category of topological spaces then cat_F is the Lusternik–Schnirelmann category cat. We are especially interested in the functor π_1: $Top \to Grp$, to the category of groups, defined on objects by $\pi_1(X) = \underset{x \in X}{*} \pi_1(X, x)$; if f: $X \to Y$ is a map, then $\pi_1(f)$: $\underset{x \in X}{*} \pi_1(X, x) \to \underset{y \in Y}{*} \pi_1(Y, y)$ restricted to $\pi_1(X, x)$ is the homomorphism f_*: $\pi_1(X, x) \to \pi_1(Y, f(x))$ induced by f on fundamental groups. One can see that a subset U of X is π_1-contractible if and only if every loop in U is contractible in X. We are also interested in the functor H_1: $Top \to Ab$ where $H_1(X)$ is the first singular homology group of X.

Proposition 2.1 (ii), with the additional hypothesis of semilocal 1-connectedness of X, is due to Ganea and Eilenberg ([5, Prop. 3], [9, Prop. 1.18]). Our results are based on it.

A *paracompact* space is a Hausdorff space such that every open cover has a locally finite refinement [17, page 156].

PROPOSITION 2.1. *Let X be a paracompact, locally pathwise connected space and let n be a natural number.*

(i) *In order that $cat_{H_1} X \leq n$ it is necessary and sufficient that there exist a complex L of dimension less than n and a map f: $X \to L$ such that f_*: $H_1 X \to H_1 L$ is an isomorphism.*

(ii) *If X is connected, in order that $cat_{\pi_1} X \leq n$ it is necessary and sufficient that there exist a connected complex L of dimension less than n and a map f: $X \to L$ such that f_*: $\pi_1 X \to \pi_1 L$ is an isomorphism.*

Proof. First, we prove the sufficiency of the conditions in (i). Suppose f: $X \to L$ induces an isomorphism of first homology groups and L is a complex of dimension less than n. Then $cat_{H_1} L \leq n$, that is, L can be covered with open sets $A_1, A_2, \ldots, A_n$ that are H_1-contractible in L. Then $f^{-1}(A_1), \ldots, f^{-1}(A_n)$ are H_1-contractible in X and so $cat_{H_1} X \leq n$.

To prove the necessity in (i) let $\{U_1, \ldots, U_n\}$ be an open cover of X where U_i is H_1-contractible in X ($i = 1, \ldots, n$). Let $\{\rho_i\}_{1 \leq i \leq n}$ be a partition of unity such that support $\rho_i \subset U_i$ ($i = 1, \ldots, n$).

For S nonempty contained in $\{1, \ldots, n\}$ define

$$W_S = \{x \in X \mid \rho_i(x) > \rho_j(x) \text{ and } \rho_i(x) > 0 \text{ for } i \in S \text{ and } j \notin S\}.$$

Let $v = \{V_j\}_{j \in J}$ be the family of components of the sets W_S. Let L be the nerve of v. A point of L will be denoted by a convex linear combination of the vertices of L. Notice that $W_S \subset U_i$ if $i \in S$. Now, if $W_S \cap W_{S'} \neq \emptyset$ then $S \cap S'$ is either S or S' and $W_S \cup W_{S'} \subset U_i$ for $i \in S \cap S'$. Hence, if $V_{j_1} \cap V_{j_2} \neq \emptyset$, then $V_{j_1} \cup V_{j_2}$ is contained in some U_i. Thus $V_{j_1} \cup V_{j_2}$ is H_1-contractible for $j_1, j_2 \in J$. Also, if $V_{j_1} \cap V_{j_2} \neq \emptyset$ and $V_{j_1} \neq V_{j_2}$ then $|S_1| \neq |S_2|$, where $V_{j_i} \subset W_{S_i}$, $i = 1, 2$. Hence, no point of X belongs to n different members of v and so, $\dim L \leq n - 1$.

If $x \in W_S$ we denote by $W_x(S)$ the vertex corresponding to the component of W_S containing x. If S is not empty and contained in $\{1, \ldots, n\}$ define p_S: $X \to R$ by

$$p_S(x) = \max\left\{\min_{i \in S} \rho_i(x) - \max_{j \notin S} \rho_j(x), 0\right\}$$

where $\max_{j \notin S} \rho_j(S)$ is taken to be 0 when $S = \{1, \ldots, n\}$. The mapping p_S is positive on W_S and zero on $X - W_S$. Let

$$\Phi_S(x) = \frac{p_S(x)}{\sum_{\emptyset \neq T \subset \{1, \ldots, n\}} p_T(x)}.$$

Now define the map f: $X \to L$ by $f(x) = \sum_S \Phi_S(x) W_x(S)$ where S runs over the subsets of $\{1, \ldots, n\}$ such that $x \in W_S$. Notice that, for every $j \in J$, $f^{-1}(st(v_j)) = V_j$ where v_j is the vertex associated to V_j.

We now prove that if X' is a component of X and L' is the component of L containing $f(X')$ then $f|X'$: $X' \to L'$ induces an epimorphism of fundamental groups. This implies that f_*: $H_1(X) \to H_1(L)$ is an epimorphism.

Take a point $*$ in X' and its image under f, also denoted by $*$, as base points for $\pi_1 X'$ and $\pi_1 L'$. Let α be a loop of L' based on $*$. It is clear that α is a product $\alpha_1 \cdots \alpha_m$ of paths such that, for $i = 1, \ldots, m$, the path α_i is contained in $st(v_{j_i})$ for some index j_i. Then $V_{j_i} \cap V_{j_{i+1}} \neq \emptyset$ for $i = 1, \ldots, m - 1$. Notice that the base point of X' is in V_{j_1}. For $i = 1, \ldots, m - 1$ take a point $x_i \in V_{j_i} \cap V_{j_{i+1}}$. Also define $x_0 = x_m = *$.

Let β_i be a path in V_{j_i} from x_{i-1} to x_i ($i = 1, \ldots, m$) and let $\beta = \beta_1 \beta_2 \cdots \beta_m$. We claim that $f\beta$ and α represent the same element of $\pi_1(X', *)$. Notice that $f(\beta_i)$ is a path in $st(v_{j_i})$.

For $i = 1, \ldots, m - 1$ let γ_i be a path in $st(v_{j_i}) \cap st(v_{j_{i+1}})$ from the terminal point of $f\beta_i$ to th terminal point of α_i. Also let γ_0 and γ_m be the constant map with image $*$. Since $\gamma_{i-1}^{-1} \cdot f\beta_i \cdot \gamma_i \cdot \alpha_i^{-1}$ is a path in $st(v_{j_i})$, it is nullhomotopic and so $\gamma_{i-1}^{-1} \cdot f\beta_i \cdot \gamma_i \sim \alpha_i$. Therefore

$$f\beta = \prod_{i=1}^{m} f\beta_i \sim \prod_{i=1}^{m} \gamma_{i-1}^{-1} \cdot f\beta_i \cdot \gamma_i \sim \prod \alpha_i = \alpha.$$

Hence $[\alpha] = f_*[\beta]$ in $\pi_1 X'$. This proves that $(f|X')_*$: $\pi_1 X' \to \pi_1 L'$ is surjective.

We now prove that f_* is a monomorphism. Suppose that $f_*([\alpha]) \in H_1(L)$ is trivial, where $[\alpha] \in H_1(X)$ is represented by a map α from an oriented 1-sphere S into X. Then $f\alpha$ can be extended to a map β: $F^2 \to L$ where F^2 is a compact oriented 2-manifold, with $\partial F^2 = S$ and the orientation of S is induced from that of F^2. Let K be a triangulation of F^2 such that the image, under β, of any 2-simplex τ of K is contained in the star of some vertex of L. For every such τ choose an index $j(\tau)$ such that $\beta(\tau) \subset st(v_{j(\tau)})$. If σ is a 1-simplex of K not contained in ∂F^2, let E_σ be the closed star of the baricenter of σ in K'', the second barycentric subdivision of K. Also, if τ is a 2-simplex of K, let E_τ be a 2-disk in the interior of τ that does not intersect any E_σ. We will define an extension $\hat{\beta}$: $\overline{F^2 - \bigcup_\sigma E_\sigma - \bigcup_\tau E_\tau} \to X$ of α such that $\hat{\beta}|\partial E_\sigma$ and $\hat{\beta}|\partial E_\tau$, are nullhomologous for any σ and any τ. If u is a vertex of K not lying on ∂F and $\tau_1, \ldots, \tau_r$ are the 2-simplexes of K containing u define $\hat{\beta}(u)$ to be a point of the nonempty set $V_{j(\tau_1)} \cap \ldots \cap V_{j(\tau_r)}$. Also, if u is any vertex of K and if x lies in a 1-simplex of K'' containing u, contained in K and not contained in ∂F, we define $\hat{\beta}(x) = \hat{\beta}(u)$. Now if σ is a 1-simplex of K not contained in ∂F we extend $\hat{\beta}$, already defined in $\partial E_\sigma \cap \sigma$, to ∂E_σ in such a way that $\hat{\beta}(\tau \cap \partial E_\sigma) \subset V_{j(\tau)}$ for any 2-simplex τ of K. This is possible since $V_{j(\tau)}$ is path-connected. Finally, if τ is a 2-simplex of K then $\overline{\tau - \cup E_\sigma - E_\tau}$ is an annulus A and $\hat{\beta}(\partial A - \partial E_\tau) \subset V_{j(\tau)}$ and so we can extend $\hat{\beta}$ to A in such a way that $\hat{\beta}(A) \subset V_{j(\tau)}$. We have, therefore, defined an extension $\hat{\beta}$: $\overline{F^2 - \bigcup_\sigma E_\sigma - \bigcup_\tau E_\tau} \to X$ of α in such a way that $\hat{\beta}(\partial E_\tau) \subset V_{j(\tau)}$ for any 2-simplex τ, and $\hat{\beta}(\partial E_\sigma) \subset V_{j(\tau_1)} \cup V_{j(\tau_2)}$ where $\tau_1 \cap \tau_2 = \sigma$. Since $V_{j(\tau)}$ and $V_{j(\tau_1)} \cup V_{j(\tau_2)}$ are H_1-contractible, $\hat{\beta}\left|\left(\bigcup_\sigma \partial E_\sigma\right) \cup \left(\bigcup_\tau \partial E_\tau\right)\right.$ is homologically trivial and, therefore, α is homologically trivial. Hence f_* is a monomorphism. This completes the proof of (i).

The proof of the sufficiency of the condition in (ii) is analogous to that of (i).

To prove necessity in (ii) let $\{U_1, \ldots, U_n\}$ be an open cover of X such that U_i is π_1-contractible in X $(i = 1, \ldots, n)$. Let $\{\rho_i\}_{1 \le i \le n}$ be a partition of unity such that support $\rho_i \subset U_i$ $(i = 1, \ldots, n)$. Define $v = \{V_j\}_{j \in J}$ and f: $X \to L$ exactly as in the proof of (i). Then $\dim L < n$, L is connected and $V_{j_1} \cup V_{j_2}$ is π_1-contractible in X if $j_1, j_2 \in J$. As shown in the proof of (i), f_*: $\pi_1 X \to \pi_1 L$ is surjective.

To prove that f_*: $\pi_1 X \to \pi_1 L$ is a monomorphism suppose that $f_*([\alpha])$ is trivial, where $[\alpha] \in \pi_1(X)$ is represented by a map α from the 1-sphere S into X. Then $f\alpha$ can be extended to a map β from a 2-disk F^2 into L. Proceed as in the proof that f_*: $H_1 X \to H_1 L$ is a monomorphism in (i) to extend α to a map $\hat{\beta}$: $\overline{F^2 - \bigcup_\sigma E_\sigma - \bigcup_\tau E_\tau} \to X$ where the E_σ and E_τ are defined as in the proof of (i). Now $\hat{\beta}(\partial E_\tau) \subset V_{j(\tau)}$ for any τ and $\hat{\beta}(\partial E_\sigma) \subset V_{j(\tau_1)} \cap V_{j(\tau_2)}$ where $\tau_1 \cap \tau_2 = \sigma$. Since $V_{j(\tau)}$ and $V_{j(\tau_1)} \cup V_{j(\tau_2)}$ are π_1-contractible in X, $\hat{\beta}$ can be extended to a map $\hat{\beta}$: $F^2 \to X$. Since $\hat{\beta}|\partial F^2 = \alpha$, $[\alpha]$ is trivial. Hence f_*: $\pi_1 X \to \pi_1 L$ is a monomorphism. □

§3. HOMOLOGY OF 3-MANIFOLD GROUPS

Another ingredient in the proof of our results is the study of the natural homomorphism $H_3(M^3; A) \to H_3(\pi_1 M^3; A)$ which we undertake in this section. As a by-product we calculate the homology of closed 3-manifold groups (see the table at the end of the section).

If M is a 3-manifold we denote by $\hat{M}$ the manifold obtained from M by capping off each 2-sphere component of ∂M with a 3-disk.

LEMMA 3.1. *Let M^3 be a prime closed 3-manifold. Let A be $\mathbf{Z}$ if M^3 is orientable and $\mathbf{Z}_2$ if M^3 is nonorientable. Let g: $M^3 \to BG$ be the natural map from M^3 to the classifying space of the fundamental group G of M^3. If $g_*: H_3(M^3; A) \to H_3(BG; A)$ is zero, then G is trivial or infinite cyclic.*

Proof. Suppose M^3 is orientable. If G were infinite noncyclic then we could take g to be the identity and g_* would be nonzero. If G were nontrivial of finite order d, then g_* would be a surjection onto a cyclic group of order d. (Compare [32; §2] or [4; chapter XIII]). Hence G is either $\mathbf{Z}$ or 1.

Next, suppose that M^3 is nonorientable (and therefore G is infinite). To complete the proof we will show that, if $G \not\approx \mathbf{Z}$ (that is, if $M^3 \not\approx S^1 \tilde{\times} S^2$), then

$$g_*: H_3(M^3; \mathbf{Z}_2) \to H_3(BG; \mathbf{Z}_2)$$

is nonzero.

If $M^3 \sim \mathbf{P}^2 \times S^1$ then $BG = \mathbf{P}^\infty \times S^1$, $g = (j \times id)h$ where j: $\mathbf{P}^2 \to \mathbf{P}^\infty$ is the natural inclusion, h: $M^3 \to \mathbf{P}^2 \times S^1$ is a homotopy equivalence and therefore $g_* \neq 0$.

Henceforth we will assume that $M^3 \nsim \mathbf{P}^2 \times S^1$. Notice that M^3 is irreducible. If M^3 contains no two-sided projective plane, then by the projective plane theorem ([6], [13, Theorem 4.12]), $\pi_2 M^3 = 0$, we can take g to be the identity and so $g_* \neq 0$. If M^3 contains a two-sided projective plane, let $\{P_1^2, \ldots, P_k^2\}$ be a maximal collection of pairwise disjoint, two-sided $\mathbf{P}^2$'s in M^3 such that no pair cobounds a homotopy $\mathbf{P}^2 \times I$ [13, Lemma 13.2]. Let N^3 be M^3 cut along $\bigcup_{i=1}^{k} P_i^2$. Notice that no component of N is a homotopy $\mathbf{P}^2 \times I$ (here we use the fact that $M \nsim \mathbf{P}^2 \times S^1$). Hence, by a theorem of Epstein ([6], [13, Theorem 9.6]) every component of N^3 has infinite fundamental group. Let $\tilde{N}^3$ be the orientable double covering of N.

By the proof of [31, Lemma 2.1], $\hat{\tilde{N}}$ is aspherical (see [31, Proposition 2.2 (c)]).

Now choose k copies $P_1^\infty, \ldots, P_k^\infty$ of $\mathbf{P}^\infty$ and identify a standard copy of $\mathbf{P}^2 \times I$ in $P_i^\infty \times I$ with a regular neighbourhood of P_i^2 in M^3 to obtain a complex $X = M \cup P_1^\infty \times I \cup \ldots \cup P_k^\infty \times I$. We will show that X can be taken as BG. First the inclusion of M in X induces an isomorphism of fundamental groups since $\pi_1 P_i^2 \to \pi_1 P_i^\infty$ is an isomorphism ($i = 1, \ldots, k$). Let $C_1, \ldots, C_r$ be the components of $X - \bigcup_{i=1}^{k} P_i^\infty \times \mathring{I}$. A C_j is obtained from a component W of N by attaching to W copies of $\mathbf{P}^\infty$ one for each component of ∂W; its orientable double cover $\tilde{C}_j$ is obtained by attaching to $\tilde{W}$ copies of S^∞ along each (spherical) component of $\partial \tilde{W}$. Hence $\tilde{C}_j \sim \hat{\tilde{W}}$ and so C_j is aspherical. Notice also that the inclusion of a copy of $\mathbf{P}^\infty$ (in $\mathrm{Fr}C_j$) in C_j induces a monomorphism of fundamental groups. We therefore have a graph of aspherical spaces, as described in [29, pp. 155–156], whose total space is X and by [29, Proposition 3.6(ii)] X is aspherical. Hence we can take X as BG and the inclusion of M in X as g. The Mayer–Vietoris sequence

$$0 \to H_3(M; \mathbf{Z}_2) \oplus H_3\left(\bigcup_{i=1}^{k} P_i^\infty \times I\right) \xrightarrow{(g_*, j_*)} H_3(X; \mathbf{Z}_2)$$

show that $g_* \neq 0$. □

Remark. The construction of the space X $(= B\pi_1 M)$ in the proof of Lemma 3.1 allows one to complete the description of the homology of a (closed) 3-manifold group. Such a group can be expressed uniquely as the free product of indecomposable 3-manifold groups and $H_n(\ast_i G_i) = \bigoplus_n H_n(G_i)$ for $n > 0$. Also $H_*(\mathbf{Z})$ is well known. We therefore describe the integral homology group $H_n(\pi_1 M^3)$ only when M^3 is irreducible.

$H_n(\pi_1 M^3)$ for M^3 closed irreducible

M^3	$n = 2$	$n = 3$	n even $n > 2$	$n \equiv 1 \bmod 4$ $n > 1$	$n \equiv 3 \bmod 4$ $n > 3$
with finite π_1	0	$\mathbf{Z}_{\lvert\pi_1 M^3\rvert}$	0	$H_1(\pi_1 M^3)$	$\mathbf{Z}_{\lvert\pi_1 M^3\rvert}$
orientable with infinite π_1	$\mathbf{Z}^r$	$\mathbf{Z}$	0	0	0
with $\pi_1 \approx \mathbf{Z} \times \mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$
nonorientable and with $\pi_1 \not\approx \mathbf{Z} \times \mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}^{r-1}$	$\mathbf{Z}_2^k$	0	$\mathbf{Z}_2^k$	$\mathbf{Z}_2^k$

$r = \dim_{\mathbf{Q}} H_1(\pi_1 M^3; \mathbf{Q})$
k = number of conjugacy classes of elements of order 2 in $\pi_1 M^3$

In connection with the last line of the table one can show that, if M^3 is closed nonorientable and irreducible, then the number of conjugacy classes of order 2 in $\pi_1 M^3$ equals the cardinality of a maximal family $\{P_1^2, \ldots, P_k^2\}$ of two-sided projective planes in M^3 such that no pair cobounds a homotopy $\mathbf{P}^2 \times I$. To see this one uses [31, Theorem 4.1], the first remark after [29, Theorem 3.7] and [13, Theorem 9.8].

§4. CATEGORIES OF 3-MANIFOLDS

Theorem 4.1 is our main result. It allows us to compute the category of any closed 3-manifold (see Corollary 4.2).

THEOREM 4.1. *Let M^3 be a closed 3-manifold. The following statements are equivalent:*

(i) $cat M^3 \leq 3$
(ii) $cat_{\pi_1} M^3 \leq 3$
(iii) $\pi_1 M^3$ *is free*

Proof. (i) $\Rightarrow$ (ii) Obvious.

(ii) $\Rightarrow$ (iii) By Proposition 2.1 there is a connected 2-complex L^2 and a map $f\colon M^3 \to L^2$ inducing isomorphism of fundamental groups. Write $G = \pi_1 M^3$. Let g be the natural map from M^3 to BG, the classifying space of G. Let $h\colon L^2 \to BG$ be a map such that hf and g induce the same homomorphism of fundamental groups. Then hf and g are homotopic and so, for any group of coefficients A, the diagram

$$\begin{tikzcd} H_3(M^3; A) \arrow[rr, "g_*"] \arrow[dr, "f_*"'] & & H_3(BG; A) \\ & H_3(L^2; A) \arrow[ur, "h_*"'] & \end{tikzcd}$$

commutes and $H_3(L^2; A) = 0$. Hence $g_* = 0$.

Now let M_1 be any prime summand of M and let $g_1\colon M_1 \to BG_1$ be the natural map where $G_1 = \pi_1 M_1$. We have a commutative diagram

$$\begin{tikzcd} H_3(M; A) \arrow[r, "g_*"] \arrow[d, "c_*"'] & H_3(BG; A) \arrow[d] \\ H_3(M_1; A) \arrow[r, "g_{1*}"] & H_3(BG_1; A) \end{tikzcd}$$

where the left vertical homomorphism is induced by a collapsing map c.

Assume that M is orientable and take $A = \mathbf{Z}$. Then c_* is surjective and, therefore $g_{1*} = 0$. Hence, by Lemma 3.1, $\pi_1 M_1$ is trivial or infinite cyclic. This proves (iii) in case M^3 is orientable.

Assume now that M^3 is nonorientable, $A = \mathbf{Z}_2$ and M_1 is any nonorientable prime summand of M^3. Then, again, c_* is surjective, $g_{1*} = 0$ and, by Lemma 4.1, $\pi_1 M_1 \approx \mathbf{Z}$, that is, $M_1 \approx S^1 \tilde{\times} S^2$. Therefore $M^3 \approx S^1 \tilde{\times} S^2 \# N^3$ where N^3 is orientable. Let $\rho\colon \tilde{M}^3 \to M^3$ be the orientable double covering and let $\{U_1, U_2, U_3\}$ be an open cover of M^3 with U_i π_1-contractible ($i = 1, 2, 3$). Then $\{\rho^{-1}(U_1), \rho^{-1}(U_2), \rho^{-1}(U_3)\}$ is an open cover of $\tilde{M}^3$ and $\rho^{-1}(U_i)$ is π_1-contractible ($i = 1, 2, 3$). Hence $cat_{\pi_1} \tilde{M}^3 \leq 3$ and, since we already know that (ii) $\Rightarrow$ (iii) in the orientable case, $\pi_1 \tilde{M}^3$ is free. Since $\tilde{M}^3 \approx N^3 \# S^1 \times S^2 \# (-N^3)$, $\pi_1 N^3$, and therefore $\pi_1 M^3$, is free.

(iii) $\Rightarrow$ (i) If $\pi_1 M^3$ is free then M^3 is homotopy equivalent to a connected sum of copies of $S^1 \times S^2$ or $S^1 \tilde{\times} S^2$ ([13, Chapter V]). Since this connected sum can be covered with three open balls ([10]), it follows that $cat M^3 \leq 3$. □

Remark. One may conjecture that, for a closed n-manifold M^n with $n \geq 3$, $cat(M^n) \leq 3$ implies that $\pi_1(M^n)$ is free.

Theorem 4.1 enables us to calculate $cat M^3$ and $cat_{\pi_1} M^3$ for any closed 3-manifold.

COROLLARY 4.2. *Let M^3 be a closed 3-manifold. Then*

$$cat M^3 = \begin{cases} 2 & \textit{if } \pi_1 M^3 = 1 \\ 3 & \textit{if } \pi_1 M^3 \textit{ is free nontrivial} \\ 4 & \textit{if } \pi_1 M^3 \textit{ is not free} \end{cases}$$

$$cat_{\pi_1} M^3 = \begin{cases} 1 & \textit{if } \pi_1 M^3 = 1 \\ 2 & \textit{if } \pi_1 M^3 \textit{ is free nontrivial} \\ 4 & \textit{if } \pi_1 M^3 \textit{ is not free.} \end{cases}$$

Proof. Since $cat_{\pi_1} M^3 \leq cat M^3 \leq 4$, it follows from Theorem 4.1 that, if $\pi_1 M^3$ is not free, then $cat_{\pi_1} M^3 = cat M^3 = 4$.

If $\pi_1 M^3 = 1$ then clearly $cat_{\pi_1} M^3 = 1$ and, since M^3 minus a 3-disk is a homotopy 3-disk and M^3 is not contractible, $cat M^3 = 2$.

Suppose $\pi_1 M^3$ is free of rank $r > 0$. Then

$$M^3 \approx \Sigma^3 \# \left(\mathop{\#}_{i=1}^{m} S^2 \times S^1 \right) \# \left(\mathop{\#}_{j=1}^{n} S^2 \tilde{\times} S^1 \right)$$

($m \geq 0$, $n \geq 0$, Σ^3 a homotopy sphere) and therefore we can find r disjoint spheres $S_1^2, \ldots, S_r^2$ in M such that $M^3 - \bigcup_{k=1}^{r} S_k^2$ is 1-connected. Since every component of a regular neighborhood of $\bigcup_{k=1}^{r} S_k^2$ is also 1-connected, $cat_{\pi_1} M^3 \leq 2$. As $\pi_1 M^3 \neq 1$, $cat_{\pi_1} M^3 = 2$. (See also [7, Theorem 23.1]). Also M^3 can be covered with three homotopy 3-cells and $cat M^3 > 2$ since M^3 is not a homotopy sphere (see [16, page 336]). Hence $cat M^3 = 3$. □

Remarks. As a consequence of the corollary one can see that if M^3 is a closed PL 3-manifold with $cat M^3 = k$ then M^3 can be covered with k subpolyhedra contractible in themselves. Thus, with the terminology of [26] and [27], the category, the geometric category and the strong category of M^3 are the same.

Denote by $(k; r)$, $r \in \mathbf{Q}$, the manifold obtained by r-surgery on the knot k of S^3. Then it follows from Theorem 4.1 and Gabai's theorem [8], that $cat(k; r) = 4$ $\forall r \in \mathbf{Q}$ if and only if k has property P. Thus the conjecture that every nontrivial knot has property P is equivalent to the conjecture that $cat(k; r) = 4$ if k is nontrivial and $r \in \mathbf{Q}$.

Two invariants of a topological manifold M^n related to $cat(M^n)$ are: the smallest number $C(M^n)$ of open balls needed to cover M^n, and the minimal number $N_0(M^n)$ of charts (spaces homeomorphic to open sets of $\mathbf{R}^n$) needed to cover M^n. (Another invariant $F(M^3)$ of M^3, which turns out to be equal to $C(M^3)$, is defined in the introduction.) Clearly $cat(M^n) \leq C(M^n)$ and $N_0(M^n) \leq C(M^n)$. One has $C(M^n) \leq n + 1$ by [20] or [28] and frequently (perhaps always) $cat(M^n) = C(M^n)$ (see [30], [25], question 9 in [26], and Conjecture 7.1 in [27]). It was proven in [11] that for a closed 3-manifold M^3, $N_0(M^3)$ is two if the Bockstein of the first Stiefel–Whitney class of M^3, $\beta\omega_1(M^3) \in H^2(M^3; \mathbf{Z}_2)$, is zero and three if it is not. Hempel and Macmillan proved that $C(M^3) \leq 3$ if and only if $\pi_1(M^3)$ is free and M^3 contains no fake cells [14]. This is analogous to our Theorem 4.1. By these two results $cat(M^3) = C(M^3)$ if and only if the Poincaré conjecture is true; in fact $catM^3 = C(M^3)$ if and only if M^3 contains no fake cells or $\pi_1 M^3$ is not free. The values of the pair $(N_0(M^3), cat(M^3))$ as M^3 runs over the class of closed 3-manifolds are (2,2), (2,3), (2,4) and (3,4); in particular $N_0(M^3) < cat(M^3)$ if M^3 is not a homotopy sphere.

Considering, again, the manifolds obtained by surgery on knots, it follows from [14], [8] and [12] that $C(k; r) = 4$ if k is a nontrivial knot and $r \in \mathbf{Q}$.

It has been conjectured ([26] Question 10, and [27], §7) that if M^n is a closed manifold then the category of M^n minus a point equals $cat(M^n) - 1$. We now prove this conjecture for the case $n = 3$.

COROLLARY 4.3. *Let M^3 be a closed 3-manifold and let $p \in M^3$. Then $cat(M^3 - p) = cat(M^3) - 1$.*

Proof. Suppose first that $\pi_1 M^3$ is not free. Then $catM^3 = 4$ and, since $M^3 - p$ is homotopy equivalent to a 2-complex and $\pi_1(M^3 - p)$ is not free, $cat(M^3 - p) = 3$ ([2]).

Next assume that $\pi_1 M^3$ is free and non trivial. Then $catM^3 = 3$ and, since M^3 is the connected sum of a homotopy 3-sphere and S^2-bundles over S^1, $M^3 - p$ can be covered with 2 homotopy cells and so $cat(M^3 - p) = 2$.

Finally, if $\pi_1 M^3$ is trivial then $cat(M^3) = 2$ and $cat(M^3 - p) = 1$. □

Remark. In contrast, it is possible to prove that, if $M^3 \not\approx S^3$, $N_0(M^3 - p) = N_0(M^3)$.

§5. CATEGORIES OF QUOTIENTS OF SPHERES

Our methods can be used to prove a theorem of Krasnoselski ([18]) stating that a manifold properly covered by the n-sphere has category $n + 1$. We give a proof in this section.

THEOREM 5.1. *Let X be a connected space of the homotopy type of a CW-complex. Let n be a natural number such that $\pi_i(X) = 0$ for $1 < i < n$ and $\pi_n X \to H_n X$ is not surjective. Then $cat_{\pi_1} X \geq n + 1$.*

Proof. We may assume X is a connected CW-complex. Let $G = \pi_1 X$. Kill $\pi_n X$ by attaching $(n+1)$-cells to X by maps $\varphi_j \colon \partial D_j^{n+1} \to X$ such that $\{[\varphi_j]\}$ generate $\pi_n X$. Then, as usual, we construct the classifying space BG, killing successively $\pi_{n+1}, \pi_{n+2}, \ldots$ by attaching cells of dimensions $n+2, n+3, \ldots$. The group $H_n BG$ is isomorphic to the cokernel of the Hurewicz homomorphism $\pi_n X \to H_n X$, and so $H_n BG \neq 0$. Moreover the inclusion induced homomorphism $g_* \colon H_n X \to H_n BG$ is surjective and, therefore, nonzero.

Now suppose $cat_{\pi_1}(X) \leq n$. Recall that a CW-complex is paracompact and locally connected [21, Theorem II.4.2 and Corollary II.6.7]. Then by [5, Proposition 3] (see also Prop. 2.1) there is a connected $(n-1)$-complex L^{n-1} and a map $f \colon X \to L^{n-1}$ inducing isomorphism of fundamental groups. Let $h \colon L^{n-1} \to BG$ be a map such that hf and g induce th same homomorphisms of fundamental groups. Then hf and g are homotopic and so, the diagram

$$\begin{tikzcd} H_n(X) \arrow[rr, "g_*"] \arrow[dr, "f_*"'] & & H_n(BG) \\ & H_n(L^{n-1}) \arrow[ur, "h_*"'] & \end{tikzcd}$$

commutes, which is impossible because $H_n(L^{n-1}) = 0$ and g_* is non-zero. Hence $cat_{\pi_1}(X) \geq n+1$. □

Krasnoselki's theorem, stating that $cat(S^n/G) = n+1$ for a free action of a finite nontrivial group G in the sphere S^n, is a consequence of Theorem 5.1:

COROLLARY 5.2. *If G is a finite nontrivial group acting freely on a homotopy sphere Σ^n, then $cat_{\pi_1}(\Sigma^n/G) = cat(\Sigma^n/G) = n+1$.*

Proof. We have $cat_{\pi_1}(\Sigma^n/G) \leq cat(\Sigma^n/G) \leq n+1$ since Σ^n/G is an n-manifold. Also Σ^n/G has the homotopy of a CW-complex ([24] or [21, Corollary IV.5.7]). If the action of G preserves the orientation then $\pi_i(\Sigma^n/G) = 0$ for $1 < i < n$, the image of a generator of $\pi_n(\Sigma^n/G)$ in $H_n(\Sigma^n/G)$ is $|G|$ times a generator of the infinite cyclic group and so, by Theorem 5.1, $n+1 \leq cat_{\pi_1}(\Sigma^n/G)$.

If the action of G does not preserve the orientation then Σ^n/G is homotopy equivalent to an even dimensional projective space [19, IV.3.1] and it is easy to show (applying $H_n(\;\;;\mathbf{Z}_2)$) that the natural map $g \colon \Sigma^n/G \to BG$ cannot be factored through an $(n-1)$-complex and so $n+1 \leq cat_{\pi_1}(\Sigma^n/G)$ again. □

Krasnoselski's proof of his result is complicated. James ([16, p. 334]) asked for a proof more in the spirit of algebraic topology. Marzantowicz ([23]) provided one such proof and we have given another one above. We now give a brief proof in case G has even order. Let Z_2 be a subgroup of order 2 of G. Then Σ^n/G is covered by Σ^n/Z_2 which is homotopy equivalent to $\mathbf{P}^n$ [19, IV.3.1]). A well known cup product argument then shows that $cat(\Sigma^n/Z_2) = n+1$, and so, using the homotopy lifting property, we have $n+1 \geq cat(\Sigma^n/G) \geq cat(\Sigma^n/Z_2) = n+1$. This last proof also works in case G has odd order using the fact that Σ^n/Z_p (p a prime divisor of $|G|$) is homotopy equivalent to a lens space L ([3, Lemma 1]) and the less elementary fact that $cat(L) = n+1$ ([22], [1]).

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